

Simple wavelet sets in \mathbb{R}^n

Kathy D. Merrill

ABSTRACT. Wavelet sets that are finite unions of convex sets are constructed in \mathbb{R}^n , $n \geq 2$, for dilation by any expansive matrix that has a power equal to a scalar times the identity and also has all singular values greater than \sqrt{n} . In particular, we produce simple wavelet sets in any dimension for dilation by any real scalar greater than 1.

1. Introduction

A *wavelet set* relative to dilation by an expansive (all eigenvalues greater than 1 in absolute value) real $n \times n$ matrix A is a set $W \subset \mathbb{R}^n$ whose characteristic function $\mathbf{1}_W$ is the Fourier transform of an orthonormal wavelet. That is, if $\hat{\psi} = \mathbf{1}_W$, then $\{\psi_{j,k} \equiv \sqrt{|\det A|^j} \psi(A^j \cdot -k), j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. This definition is equivalent to the requirement that the set $W \subset \mathbb{R}^n$ tiles n -dimensional space (almost everywhere) both under translation by \mathbb{Z}^n and under dilation by the transpose A^* , so that

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_W(x + k) = 1 \quad a.e. \ x \in \mathbb{R}^n, \text{ and}$$

$$\sum_{j \in \mathbb{Z}} \mathbf{1}_W(A^{*j}x) = 1 \quad a.e. \ x \in \mathbb{R}^n.$$

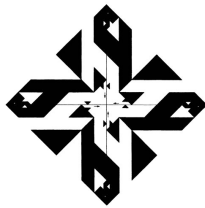
While wavelet set wavelets are not well-localized, and thus not directly useful for applications, they have proven to be an essential tool in developing wavelet theory. In particular, wavelet set examples established that not all wavelets have an associated MRA [9], and that single wavelets exist for an arbitrary expansive matrix in any dimension [8].

2010 *Mathematics Subject Classification.* Primary 42C40, 52C22.

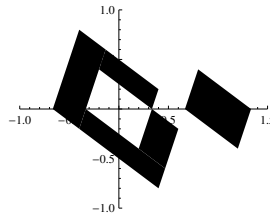
Key words and phrases. wavelet set, tiling.

Smoothing and interpolation techniques have also used wavelet set wavelets to produce more well-localized examples. (See e.g. [11], [12], [7], [6], [1], [13], [3].)

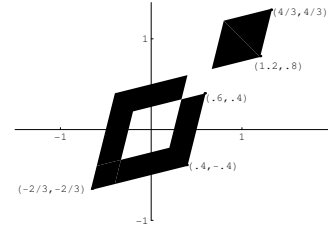
All of the early examples of wavelet sets for dilation by non-determinant 2 matrices in dimension greater than 1 were geometrically complicated, showing the fingerprints of the infinite iterated process used to construct them. (See e.g. Figure 1(a)). Many early researchers, e.g. [6], [16], conjectured that a wavelet set for dilation by 2 in dimension greater than 1 could not be written as a finite union of convex sets. In support of this conjecture, Benedetto and Sumetkijakan [6] showed that a wavelet set for dilation by 2 in \mathbb{R}^n cannot be the union of n or fewer convex sets. However, in 2004, Gabardo and Yu [10] used self-affine tiles to produce a wavelet set for dilation by 2 in \mathbb{R}^2 that is a finite union of polygons (Figure 1(b)). In 2008 [14] we used a technique based on generalized multiresolution analyses [2] to construct such wavelet sets for arbitrary real ($d > 1$) scalar dilations in \mathbb{R}^2 . Figure 1(c) shows one of the wavelet sets for dilation by 2 from [14]. Although they were developed independently, and using very different techniques, these two examples are remarkably similar. In fact, the wavelet sets in Figure 1(b) and 1(c) are equivalent in the sense that one can be transformed into the other under multiplication by a determinant 1 integer matrix. The similar shape of these two wavelet sets suggests the general n -dimensional result produced in this paper.



Soardi/Wieland 1998



Gabardo/Yu 2004



Merrill 2008

FIGURE 1. Wavelet sets for dilation by 2 in \mathbb{R}^2

We call wavelet sets that are finite unions of convex sets *simple wavelet sets*. In 2012 [15], we expanded the results in [14] to produce simple wavelet sets for dilation by any 2×2 matrix which has a positive integral power equal to a scalar times the identity, as long as its singular values are all greater than $\sqrt{2}$. In that paper, we also found examples

of expansive 2×2 matrices that cannot have simple wavelet sets. It is our conjecture that an expansive matrix whose determinant does not have absolute value equal to 2 can have a simple wavelet set if and only if it has a positive integer power equal to a scalar times the identity.

In this paper, we generalize the 2-dimensional examples in [15] to n -dimensional space, $n \geq 2$. We do this using neither the generalized multi-resolution analysis techniques of [15], nor the self-affine techniques of [10]. Rather, we use a remarkable result by Sherman Stein [17] on tiling \mathbb{R}^n by notched cubes, together with the tiling conditions that are equivalent to the definition of a wavelet set. Section 2 skews and translates Stein's notched n -cubes to produce notched parallelotopes that are wavelet sets for dilation by negative scalars. Section 3 further modifies these notched parallelotopes by translating out a central parallelotope (as in Figure 1(b) and 1(c)), thus creating wavelet sets for positive scalar dilations, and more generally for matrices that have a positive integral power equal to a scalar, as long as their singular values are not too small.

2. Tiling with notched parallelotopes

We begin by establishing some notation. Write $\{e_1, e_2, \dots, e_n\}$ for the standard basis of \mathbb{R}^n , and C for the cyclic permutation matrix with columns $(e_2, e_3, \dots, e_n, e_1)$. Let $\vec{1}$ stand for the vector $(1, 1, \dots, 1) \in \mathbb{R}^n$, and write τ_t for translation by $t \in \mathbb{R}^n$.

Given a vector v in \mathbb{R}^n that is not a multiple of $\vec{1}$, let

$$\mathcal{P}[v] = \{x_0 v + x_1 C(v) + \dots + x_{n-1} C^{n-1}(v) : 0 \leq x_i \leq 1\}$$

be the parallelotope spanned by the vectors $\{v, C(v), \dots, C^{n-1}(v)\}$. Note that $\mathcal{P}[v]$ has two vertices on the line determined by $\vec{1}$: one at the origin, and the other determined by the sum of the coordinates of v , at $(\sum v_i) \vec{1}$. Given an α , $0 < \alpha < 1$, write $\mathcal{N}[v, \alpha]$ for the notched parallelotope that results from deleting a subparallelotope scaled by α from the vertex $(\sum v_i) \vec{1}$ of $\mathcal{P}[v]$. That is, let

$$\mathcal{N}[v, \alpha] = \mathcal{P}[v] \setminus \tau_{(1-\alpha)(\sum v_i) \vec{1}} \alpha \mathcal{P}[v].$$

We will need the following result about translation tilings by notched cubes due to Sherman Stein.

LEMMA 2.1. *Given a real number $0 < \alpha < 1$, let L be the lattice spanned by the columns of $I - \alpha C$, where C is the cyclic permutation matrix. Then the translates of the notched unit cube $\mathcal{N}[e_1, \alpha]$ by the vectors in L tile \mathbb{R}^n .*

PROOF. See [17] □

We use this result to produce a notched parallelotope that tiles \mathbb{R}^n under translation by the lattice \mathbb{Z}^n :

LEMMA 2.2. *For a fixed real number α , $0 < \alpha < 1$, let $w(\alpha) = \frac{1}{1-\alpha^n}(1, \alpha, \alpha^2, \dots, \alpha^{n-1})$. Then the translates of $\mathcal{N}[w(\alpha), \alpha]$ by \mathbb{Z}^n tile \mathbb{R}^n .*

PROOF. By Lemma 2.1 we know that $\mathcal{N}[e_1, \alpha]$ tiles \mathbb{R}^n under translation by L , the lattice spanned by the columns of $I - \alpha C$. If we define A to be the linear transformation that maps L to \mathbb{Z}^n , we thus have that $A(\mathcal{N}[e_1, \alpha])$ tiles \mathbb{R}^n under translation by \mathbb{Z}^n . Note that $A = (I - \alpha C)^{-1} = \sum_{i=0}^{\infty} (\alpha C)^i = \frac{1}{1-\alpha^n} \sum_{i=0}^{n-1} (\alpha C)^i$. Thus, applying A to the notched unit cube $\mathcal{N}[e_1, \alpha]$ will result in the notched parallelotope $\mathcal{N}[w(\alpha), \alpha]$. □

To be a wavelet set, a notched parallelotope would have to also tile under dilation; that is, its inverse dilate would need to fit perfectly into its notch. As this is clearly impossible with the origin as one of the extreme points of the parallelotope, we consider translates $\tau_t \mathcal{N}[w(\alpha), \alpha]$. For the dilate of $\tau_t \mathcal{N}[w(\alpha), \alpha]$ by $\frac{1}{d}$, $d > 1$ to fit into its notch would require $\alpha = \frac{1}{d}$ and $\frac{1}{d}t = t + \vec{1}$, so that $t = -\frac{d}{d-1}\vec{1}$. Since this again forces the origin to be an extreme point of the parallelotope, tiling by dilation cannot work for positive scalar dilations. However, for negative scalar dilations, a wavelet set that is just a notched parallelotope is possible, as the following theorem shows. The examples produced by Theorem 2.3 generalize the wavelet sets for negative scalar dilations in \mathbb{R}^2 found in [10] and [15].

THEOREM 2.3. *For $d \in \mathbb{R}$, $d > 1$, let $w(\frac{1}{d}) = \frac{1}{d^n-1}(d^n, d^{n-1}, \dots, d)$, and $t = -\frac{d^2}{d^2-1}$. Then*

$$\mathcal{W} = \tau_{t\vec{1}} \mathcal{N} \left[w \left(\frac{1}{d} \right), \frac{1}{d} \right]$$

is a wavelet set for dilation by $-d$ in \mathbb{R}^n .

PROOF. We know from Lemma 2.2 that $N[w(\frac{1}{d}), \frac{1}{d}]$, and thus \mathcal{W} , tiles \mathbb{R}^n under translation by \mathbb{Z}^n . It remains to show that \mathcal{W} tiles under dilation by $-d$.

The parallelotope $\frac{-1}{d}(\tau_{t\vec{1}} P[w])$ has its vertices on the line determined by $\vec{1}$ at $(\frac{d}{d^2-1})\vec{1}$ and at $(\frac{-1}{d^2-1})\vec{1}$. Since the inside corner of the notch of $\tau_{t\vec{1}} \mathcal{N}[w(\frac{1}{d}), \frac{1}{d}]$ is at $(t+1)\vec{1} = -\frac{1}{d^2-1}\vec{1}$, this shows that $\mathcal{W} \cup -\frac{1}{d}\mathcal{W} = \tau_{t\vec{1}} \mathcal{P}[w] \setminus \frac{1}{d^2}(\tau_{t\vec{1}} P[w])$, which tiles under dilation by d^2 . Thus \mathcal{W} tiles under dilation by $-d$. \square

Figure 2 shows the wavelet set produced by Theorem 2.3 for dilation by -2 in \mathbb{R}^3 .

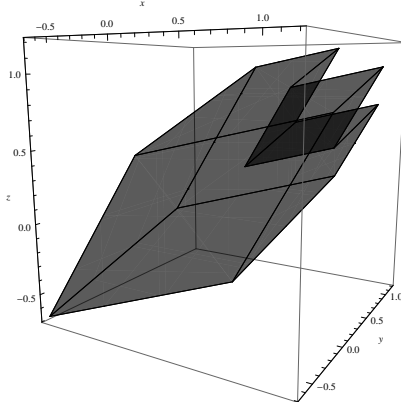


FIGURE 2. Wavelet set for dilation by -2 in \mathbb{R}^3

3. Notched parallelotopes with satellites

Even though we have seen that a translate of a notched parallelotope $\tau_t \mathcal{N}[w(\alpha), \alpha]$ cannot itself tile under dilation by a positive scalar, we will make a simple alteration to such a set that retains the property of tiling under translation, and makes the set tile under dilation as well. We use the idea behind the wavelet set construction technique in [4]. That is, we eliminate overlap between our proposed wavelet set and its dilate, by translating the dilate out by an integer vector, creating a satellite. In order to avoid the iterated process required in [4], we translate out a little bigger piece than the dilate of $\tau_t \mathcal{N}[w(\alpha), \alpha]$; that is, we translate out the dilate of the whole parallelotope $\tau_t \mathcal{P}[w]$. We choose the translation amounts such that a dilate of the satellite exactly fills the notch. The details of this construction are carried out in the following theorem.

THEOREM 3.1. *For $d \in \mathbb{R}$, $d \geq 2$, let $w(\frac{1}{d^2}) = \frac{1}{d^{2n-1}}(d^{2n}, d^{2n-2}, \dots, d^2)$. Suppose $k \in \mathbb{Z}$ satisfies $1 \leq k < d$, and let $t = \frac{d(k-d)}{d^2-1}$. Then*

$$\mathcal{W} = \left(\tau_{t\vec{1}} \mathcal{N} \left[w \left(\frac{1}{d^2} \right), \frac{1}{d^2} \right] \right) \setminus \left(\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w] \right) \cup \tau_{k\vec{1}} \left(\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w] \right)$$

is a wavelet set for dilation by d in \mathbb{R}^n .

PROOF. We claim that $\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w] \subset \tau_{t\vec{1}} \mathcal{N} \left[w(\frac{1}{d^2}), \frac{1}{d^2} \right]$. To see this, first note that $t < 0$ and $-t < \frac{d^2}{d^2-1} = \sum_{i=1}^{n-1} w_i$, so that $0 \in \frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w] \subset \tau_{t\vec{1}} \mathcal{P}[w]$. Thus, to establish the claim, it will suffice to show that the vertex of the notch that is closest to the origin, namely $(t + (1 - \frac{1}{d^2}) \sum w_i) \vec{1}$, lies outside of $\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w]$. That is, we must show that $t + 1 \geq \frac{1}{d} \left(t + \frac{d^2}{d^2-1} \right)$. Substituting $t = \frac{d(k-d)}{d^2-1}$, we see that this is equivalent to $k \geq \frac{1}{d-1}$, which follows from the given conditions $k \geq 1$ and $d \geq 2$.

Lemma 2.2 implies that $\mathcal{N} \left[w(\frac{1}{d^2}), \frac{1}{d^2} \right]$ tiles under translation by the integer lattice, and thus that $\tau_{t\vec{1}} \mathcal{N} \left[w(\frac{1}{d^2}), \frac{1}{d^2} \right]$ does as well. By the claim, we have that \mathcal{W} is formed from $\tau_{t\vec{1}} \mathcal{N} \left[w(\frac{1}{d^2}), \frac{1}{d^2} \right]$ by translating the subset $\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w]$ by an element of the integer lattice. Thus \mathcal{W} tiles \mathbb{R}^n under translation by \mathbb{Z}^n .

To establish tiling under dilation by d , we first show that $\tau_{t\vec{1}} \mathcal{N} \left[w(\frac{1}{d^2}), \frac{1}{d^2} \right]$ is disjoint from $\tau_{k\vec{1}} \left(\frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w] \right)$. That is, we must show that $\frac{t}{d} + k > t + 1$, which follows from the definition of t together with the condition $k \geq 1$. Now, note that $\tau_{t\vec{1}} \mathcal{P}[w] \setminus \frac{1}{d} \tau_{t\vec{1}} \mathcal{P}[w]$ tiles \mathbb{R}^n under dilation by d . The definition of t together with the disjointness of the pieces of \mathcal{W} shows that \mathcal{W} is formed from this set by dilating the notch $\tau_{(t+1)\vec{1}} \left(\frac{1}{d^2} \mathcal{P}[w] \right)$ by d . Thus, \mathcal{W} also tiles \mathbb{R}^n under dilation by d . \square

REMARK 3.2. The wavelet sets produced by Theorem 3.1 are a natural generalization of the 2 dimensional simple wavelet sets produced for scalar dilations in [14]. We can also alter these examples to produce natural generalizations of the 2-dimensional example for dilation by 2 that appears in [10]. Note that if \mathcal{W} is a wavelet set for dilation by the scalar d , then so is $S(\mathcal{W})$ for any integer matrix S of determinant ± 1 . If we take S to be the $n \times n$ matrix that has 1's on the diagonal, -1's on the subdiagonal and 0's elsewhere, then $S(\mathcal{W})$ is a simple wavelet set for dilation by d in dimension n that is centered on the

x_1 axis rather than the line $x_1 = x_2 = \dots x_n$. Other variations are easily produced using other choices for the matrix S .

Figure 3 below shows one of the wavelet sets produced by Theorem 3.1 for dilation by 2 in \mathbb{R}^3 , as well as a variation described in Remark 3.2.

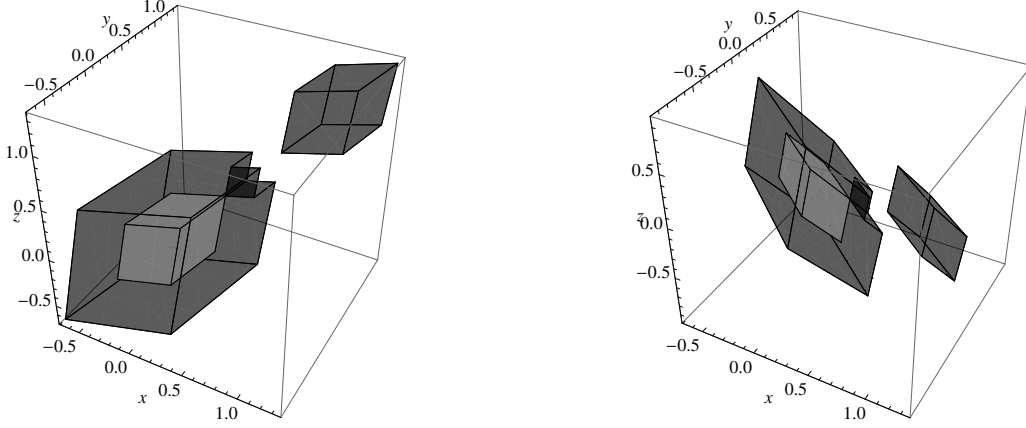


FIGURE 3. 3-dimensional wavelet sets for dilation by 2.

The final theorem uses the same technique as Theorem 3.1 to produce simple wavelet sets for dilation by expansive matrices that have a power equal to a scalar multiple of the identity, as long as their singular values are all larger than \sqrt{n} .

THEOREM 3.3. *Let A be an $n \times n$ integer matrix such that $A^p = d(I[n])$, where $d > 1$, $p \in \mathbb{Z}$, $p > 1$, and $I[n]$ is the $n \times n$ identity matrix. Suppose further that all the singular values of A are greater than \sqrt{n} . Let $w(\frac{1}{d^q}) = \frac{1}{d^{qn-1}}(d^{qn}, d^{qn-q}, \dots, d^q)$. Let k be the closest vector in \mathbb{Z}^n to $A^{*-1}(\frac{d^q}{2}\vec{1})$, and let $t = \frac{1}{d^q-1}(A^*k - d^q\vec{1})$. Then for q sufficiently large,*

$$\mathcal{W} = \left(\tau_t \mathcal{N} \left[w \left(\frac{1}{d^q} \right), \frac{1}{d^q} \right] \right) \setminus \left(A^{*-1} \tau_t \mathcal{P} \left[w \left(\frac{1}{d^q} \right) \right] \right) \cup \left(\tau_k \left(A^{*-1} \tau_t \mathcal{P} \left[w \left(\frac{1}{d^q} \right) \right] \right) \right)$$

is a wavelet set for dilation by A in \mathbb{R}^n .

PROOF. By Lemma 2.2, $\mathcal{N}[w(\frac{1}{d^q}), \frac{1}{d^q}]$, tiles under translation by \mathbb{Z}^n , and thus its translate by t does as well. Thus, \mathcal{W} will also tile by translation as long as the set $A^{*-1} \tau_t \mathcal{P}[w(\frac{1}{d^q})]$, which is translated out by k , is a subset of $\tau_t \mathcal{N}[w(\frac{1}{d^q}), \frac{1}{d^q}]$. As $q \rightarrow \infty$, the set $\tau_t \mathcal{P}[w(\frac{1}{d^q})]$ approaches the unit n -cube centered at the origin. Thus, by taking q sufficiently large, the longest vector in $\tau_t \mathcal{P}[w(\frac{1}{d^q})]$ will be arbitrarily close to \sqrt{n} times

as long as the shortest vector. Then, since the singular values of A are greater than \sqrt{n} , we will have $A^{*-1}(\tau_t \mathcal{P}[w(\frac{1}{d^q})]) \subset \tau_t \mathcal{P}[w(\frac{1}{d^q})]$. The size of the notch also shrinks to 0 as $q \rightarrow \infty$, so that by taking q larger if necessary, we will also have $A^{*-1}(\tau_t \mathcal{P}[w(\frac{1}{d^q})]) \subset \tau_t \mathcal{N}[w(\frac{1}{d^q}), \frac{1}{d^q}]$.

To establish tiling under dilation, note that because of the containment established above, $\tau_t \mathcal{P}[w(\frac{1}{d^q})] \setminus (A^{*-1} \tau_t \mathcal{P}[w(\frac{1}{d^q})])$ tiles \mathbb{R}^n under dilation by A^* . The definition of t assures that the outlier piece of \mathcal{W} exactly fits into the notch of $\tau_t \mathcal{N}[w(\frac{1}{d^q}), \frac{1}{d^q}]$ under dilation by $A^{*(-pq+1)}$. Thus we have that \mathcal{W} also tiles \mathbb{R}^n under dilation by A^* . \square

REMARK 3.4. Theorem 3.3 also produces simple wavelet set for scalar dilations $1 < d < 2$, which were not covered by Theorem 3.1. For scalar dilations by $d \geq 2$, Theorem 3.3 produces a series of alternative wavelet sets to those of Theorem 3.1. As q increases in this series, the parallelotope becomes closer to cubic, the notch becomes smaller, and the satellite becomes farther removed.

EXAMPLE 3.5. Let $A^* = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & -3 \end{pmatrix}$. Then $A^2 = 9I$ and $\text{SingularValues}(A) =$

$\{3.54, 3, 2.54\}$, so Theorem 3.3 applies. With $w(\frac{1}{9}) = (\frac{9^3}{9^3-1}, \frac{9^2}{9^3-1}, \frac{9}{9^3-1})$, $k = (1, 1, -1)$ and $t = (-\frac{3}{4}, -\frac{3}{4}, -\frac{5}{8})$, we have $A^{*-1} \tau_t \mathcal{P}[w(\frac{1}{9})] \subset \tau_t \mathcal{P}[w(\frac{1}{9})]$. Thus, we have a simple wavelet set

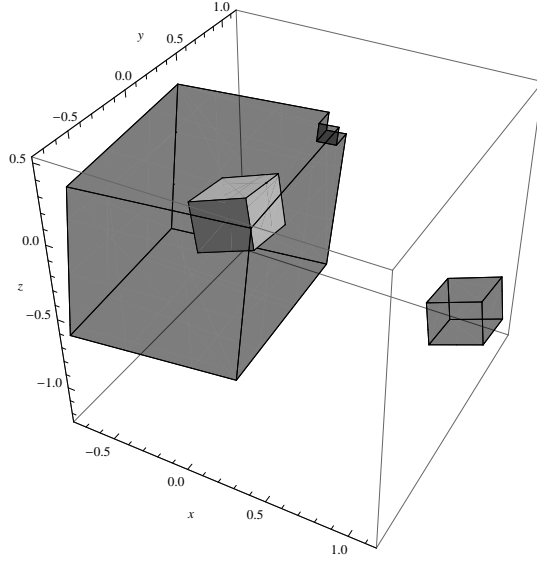
$$\mathcal{W} = \left(\tau_t \mathcal{N}\left[w\left(\frac{1}{9}\right), \frac{1}{9}\right] \right) \setminus \left(A^{*-1} \tau_t \mathcal{P}\left[w\left(\frac{1}{9}\right)\right] \right) \cup \left(\tau_{(1,1,-1)} \left(A^{*-1} \tau_t \mathcal{P}\left[w\left(\frac{1}{9}\right)\right] \right) \right),$$

which is pictured in Figure 4.

The hypothesis in Theorem 3.3 that the singular values of A be greater than \sqrt{n} is sufficient but not necessary, as the next example shows.

EXAMPLE 3.6. Let $B^* = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Then $B^2 = 4I$ and $\text{SingularValues}(B) =$

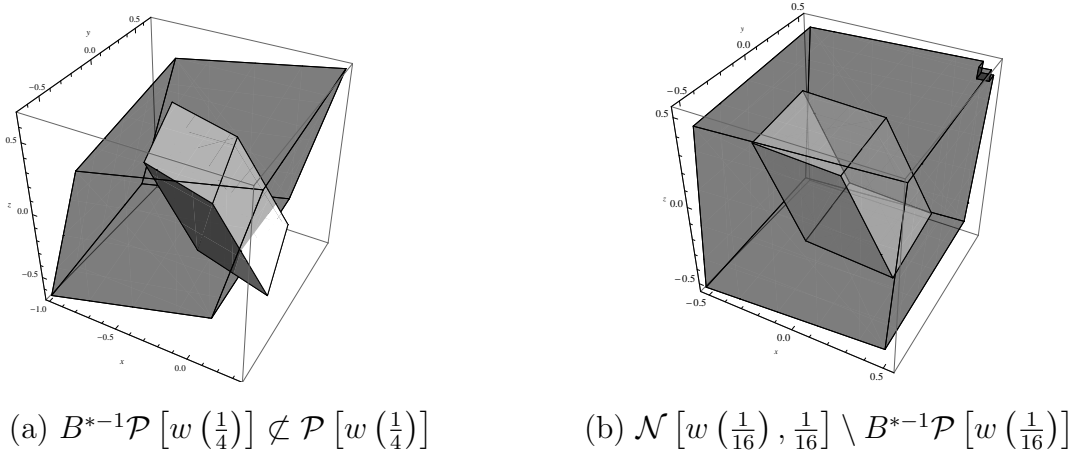
$\{2.56, 2, 1.56\}$, so Theorem 3.3 does not apply. With $w(\frac{1}{4}) = (\frac{64}{63}, \frac{16}{63}, \frac{4}{63})$, $k = (1, -1, -1)$ and $t = (-1, -\frac{2}{3}, -\frac{2}{3})$, we do not have $B^{*-1} \tau_t \mathcal{P}[w(\frac{1}{4})] \subset \tau_t \mathcal{P}[w(\frac{1}{4})]$. (See Figure 5(a).)

FIGURE 4. Simple wavelet set for the matrix A of Example 3.5.

However, using $q = 2$, with $w(\frac{1}{16}) = (\frac{2^{12}}{2^{12}-1}, \frac{2^8}{2^{12}-1}, \frac{2^4}{2^{12}-1})$, $t = -\frac{8}{15}\vec{1}$, and $k = (6, -4, -4)$, the required containment does hold, yielding the wavelet set

$$\mathcal{W} = \left(\tau_t \mathcal{N} \left[w \left(\frac{1}{16} \right), \frac{1}{16} \right] \right) \setminus \left(B^{*-1} \tau_t \mathcal{P} \left[w \left(\frac{1}{16} \right) \right] \right) \cup \left(\tau_k \left(B^{*-1} \tau_t \mathcal{P} \left[w \left(\frac{1}{16} \right) \right] \right) \right).$$

Figure 5(b) shows the central part of the wavelet set. (The complete wavelet set also includes a translation of the missing inner parallelopete by $(6, -4, -4)$.)

FIGURE 5. Building a simple wavelet set for the matrix B of Example 3.6.

References

- [1] L. Baggett, P. Jorgensen, K. Merrill, and J. Packer (2005). A non-MRA C^r frame wavelet with rapid decay. *Acta Appl. Math.* 89:251-270.
- [2] L. W. Baggett, H. A. Medina, and K. D. Merrill (1999). Generalized multi-resolution analyses and a construction procedure for all wavelet sets in \mathbb{R}^n . *J. Fourier Anal. Appl.* 5:563-573.
- [3] J. J. Benedetto and E. King (2009), Smooth functions associated with wavelet sets on \mathbb{R}^d , $d \geq 1$, and frame bound gaps, *Acta Appl. Math.*, 107:121-142.
- [4] J. J. Benedetto and M. T. Leon (1999). The construction of multiple dyadic minimally supported frequency wavelets on \mathbb{R}^d . *Contemp. Math.* 247:43-74.
- [5] J. J. Benedetto and S. Sumetkijakan (2006). Tight frames and geometric properties of wavelet sets. *Advances in Comp. Math.* 24:35-56.
- [6] M. Bownik and D. Speegle (2002). Meyer Type Wavelet Bases in \mathbb{R}^2 . *J. Approx. Th.* 116:49-75.
- [7] X. Dai and D. R. Larson (1998). Wandering vectors for unitary systems and orthogonal wavelets. *Mem. AMS* 134: No. 640.
- [8] X. Dai, D. R. Larson, and D. M. Speegle (1997). Wavelet sets in \mathbb{R}^n . *J. Fourier Anal. Appl.* 3:451-456.
- [9] I. Daubechies (1992). *Ten Lectures on Wavelets*. American Mathematical Society, Providence RI.
- [10] J-P Gabardo and X. Yu. (2004). Construction of wavelet sets with certain self-similarity properties. *J. Geom. Anal.* 14: 629-651.
- [11] E. Hernández, X. Wang, and G. Weiss (1996). Smoothing minimally supported frequency wavelets I. *J. Fourier Anal. Appl.* 2:329-340.
- [12] E. Hernández, X. Wang, and G. Weiss (1997). Smoothing minimally supported frequency wavelets II. *J. Fourier Anal. Appl.* 3:23-41.
- [13] K. D. Merrill (2008) Smooth well-localized Parseval wavelets based on wavelet sets in \mathbb{R}^2 . *Contemp. Math.* 464: 161-175.
- [14] K. D. Merrill (2008). Simple wavelet sets for scalar dilations in $L^2(\mathbb{R}^2)$. In, *Wavelets and Frames: a Celebration of the Mathematical Work of Lawrence Baggett* (P. Jorgensen, K. Merrill and J. Packer eds.), Birkhauser, Boston, pp.177-192.
- [15] K. D. Merrill (2012). Simple wavelet sets for matrix dilations in \mathbb{R}^2 . *Num. Funct. Anal. Opt.* 33:1112-1125.
- [16] P. M. Soardi and D. Weiland (1998). Single wavelets in n-dimensions. *J. Fourier Anal. Appl.* 4:299-315.
- [17] S.K. Stein (1990). The notched cube tiles \mathbb{R}^n . *Discrete Math.* 80:335-337.

KATHY MERRILL, DEPARTMENT OF MATHEMATICS, COLORADO COLLEGE, COLORADO SPRINGS,
COLORADO, 80903, USA

E-mail address: `kmerrill@coloradocollege.edu`